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§1 This week: Combinatorics

Combinatorics: Study of counting stuff

Themes:

- exploiting symmetry,
- take of advantage constraints
- testing small cases
- use metaphor.

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§ 2 Pointers paradise

<u>Setup</u>: Some finite # of people are sitting in a circle and are pointing:

a) they can only point fin. many times,

b) point to themselves,

c) can point to same person mult. times (includes e).

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Defn: A pointer's paradise is a configuration of pointing where every person is pointed at more than they're pointing.

Q: For which n (= # of people) do pointer's paradise exist?

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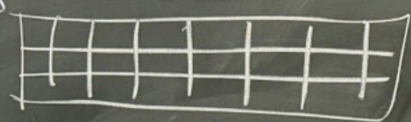
Prop: A pointer's paradise never exists.

Pf: Let $P_1, \ldots, P_n$ be the set of people. Let for each i $A_i = \#\, P_i$ is pointed at, $B_i = \#\, P_i$ is pointing. A pointer's paradise happens if $A_i > B_i$ for all i. Then $\sum_{i=1}^{n} A_i > \sum_{i=1}^{n} B_i$. But each point increases $\sum A_i$ and $\sum B_i$ by exactly 1. So $\sum A_i = \sum B_i$. Contradiction. $\blacksquare$

§3 The Chocolate bar

Set up: Have $n \times m$ bar of chocolate

3×8



A *breaking procedure* is a series of breaks of a piece of chocolate into two so that at the end I have $n \cdot m$ squares of chocolate.

Q: What is the most efficient breaking procedure (i.e., least # of steps).

Prop: All breaking procedures take the same # of steps!

Pf: Let my breaking procedure take k steps. For $i = 0, \ldots, k$ $R_i = \#$ of rectangles at that step. Note that $R_0 = 1$, but also $R_k = nm$. But, observe $R_{i+1} = R_i + 1$.

So, $nm = R_k = R_{k-1} + 1 = R_{k-2} + 2 = \cdots = R_0 + k = 1 + k$. So, $k = nm - 1$. ∎

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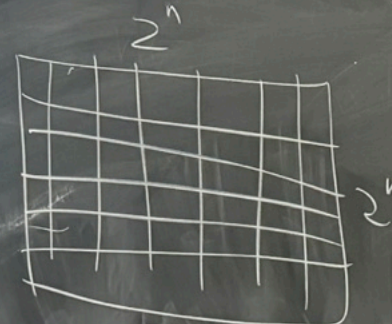
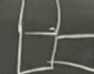
§ 4 L-Shaped tilings

Def'n: An $n \times m$ checkerboard is a collection of nm squares arranged in a grid of n rows by m colums.

Setup: $G =$ 

$\boxed{8}$ Def'n: An L-tiling of G is a filling in of G by shapes of the form  (or its rotations) which cover it w/ no overlaps.

Q: Is G L-tileable?

A: # of squares in $G = 2^n \cdot 2^n = 4^n$. If you could L-tile w/ k L's then # of squares $= 3 \cdot k$. But $4^n \neq 3k$ as $3/4^n$.

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<u>Last week</u>: We proved by induction $3 \mid 4^n - 1$.

<u>Q</u>: Is $G - \square$ L-tilable? For $n \geq 1$

<u>Prop</u>: $G - \square$ is always L-tilable.

<u>Pf</u>: We proceed by induction.

<u>Base case</u>:  is an L, $n=1$ is OK.

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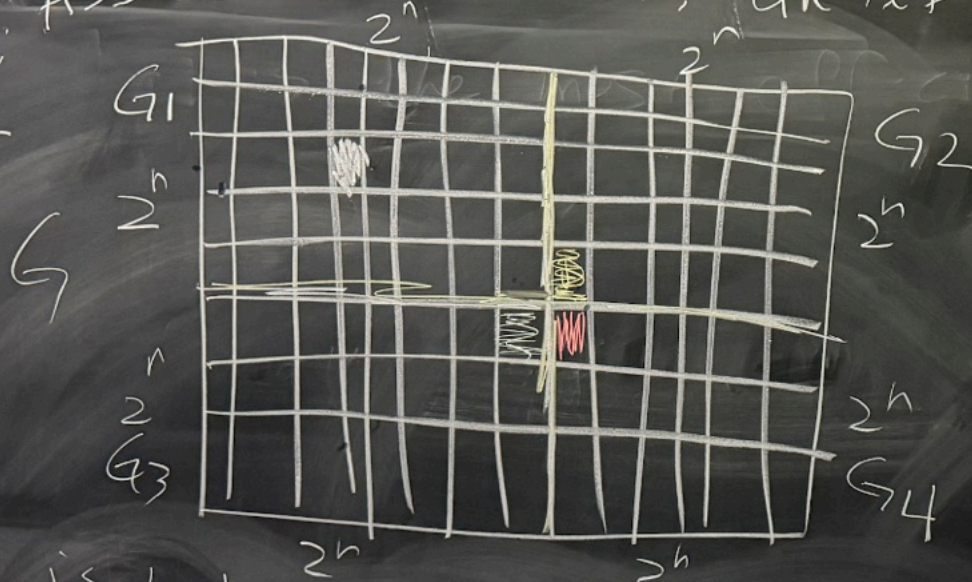
IH: Assume true for n, and let $G = 2^{n+1} \times 2^{n+1}$ checkerboard



Note $G_1 -$ is L-tilable by induction as G_1 is $2^n \times 2^n$. Similarly,

$G_2 -$, $G_3 -$, $G_4 -$ are also L-tilable by induction

So, $G -$ + , is L-tilable so $G -$ is L-tilable

by adding back in QED